On Killing vectors in initial value problems for asymptotically flat space-times

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Abstract

The existence of symmetries in asymptotically flat space-times are studied from the point of view of initial value problems. General necessary and sufficient (implicit) conditions are given for the existence of Killing vector fields in the asymptotic characteristic and in the hyperboloidal initial value problem (both of them are formulated on the conformally compactified space-time manifold).

1 Introduction

The most convenient way of considering the far fields of isolated gravitational systems is to use the conformal technique introduced by Penrose (see in [16]). In this setting one works on the conformally extended space-time manifold where points at infinity (with respect to the physical metric) are glued to the physical space-time manifold, i.e on this extended, unphysical space-time manifold they are represented by regular points. This means that one works on finite regions of the unphysical space-time, where one can use all the tools of standard, local differential geometry to perform calculations, so one avoids the determination of limits at infinity. Of course, not all type of space-times admit the construction of conformal infinities, those where the conformal extension can be performed are called asymptotically simple. Asymptotically flat are those asymptotically simple space times, where the cosmological constant vanishes. Space-times representing isolated gravitational systems are supposed to be asymptotically flat.

Several well-defined initial value problems can be formulated on the extended, unphysical space-time manifold for asymptotically flat space-times, e.g. the following initial value problems were studied extensively in the literature.

- In the asymptotic characteristic initial value problem the data are given on past null infinity $\mathcal{J}^-$ and on an incoming null hypersurface $\mathcal{N}$ which intersects $\mathcal{J}^-$ in a space-like surface $\mathcal{Z}$ diffeomorphic to $\mathcal{S}^2$ (the problem could be analogously formulated for future null infinity $\mathcal{J}^+$ with an intersecting, outgoing null hypersurface, as well).

- In the hyperboloidal initial value problem the data are given on a (3-dimensional) space-like hypersurface $\mathcal{S}$ intersecting future null infinity $\mathcal{J}^+$ in a space-like surface $\mathcal{Z}$ diffeomorphic to $\mathcal{S}^2$. The term "hyperboloidal" comes from the fact that the physical metric on $\mathcal{S}$ behaves near $\mathcal{Z}$ like that of a space with constant negative curvature. This problem is not time symmetric in the sense that the Cauchy development of a hyperboloidal hypersurface intersecting past null infinity $\mathcal{J}^-$, in opposition to the problem formulated with respect to $\mathcal{J}^+$, does not extends up to null infinity.

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These initial value problems are solved, i.e. there are proved uniqueness and existence theorems for these cases (see in [4],[15],[5], a review can be found in [6]).

- The standard Cauchy problem is still not solved completely, it is not known what kind of asymptotic conditions are necessary to be imposed on the initial data in order to get smooth null infinity in the time evolution, i.e. to get asymptotically flat space-time in the sense defined by Penrose (the state of the research is reviewed in [7]).

Working on the unphysical space-time provides some advantages even for numerical calculations, because the evolution can be calculated over a finite grid covering a conformally compactified initial space-like hypersurface (some recent results can be found in Ref. [1, 2, 3, 10, 11, 12, 13, 14]).

In this paper we want to formulate initial value problems for Killing vector fields in asymptotically flat space-times. Our results will be applicable for both of the hyperboloidal and the asymptotic characteristic initial value problems. We will formulate general (implicit) conditions on the initial data, which guarantee the existence of Killing vector fields in the time evolution. Our method is essentially the same as in [17], where the same problem was solved on the physical space-time manifold for all type of the initial value problems which could be relevant there (see also [8, 18]). However, we will study here the above introduced asymptotic characteristic and hyperboloidal initial value problems, so we have to work in terms of the conformal quantities on the extended, unphysical space-time manifold. The scheme of our proofs is essentially the same as in [17], however the corresponding calculations are much more complicated. First we will discuss the problem in general, then in the appendix two examples, space-times with Klein-Gordon and Maxwell fields, will be studied in more detail.

2   Killing fields on the unphysical space-time

In the following we will use the notation that quantities which are defined with respect to the physical space-time manifold $(\tilde{M}, \tilde{g}_{ab})$ wear a tilde, and the indices of such (tensorial) quantities will be raced and lowered by the physical metric $(\tilde{g}_{ab}, \tilde{g}^{ab})$.

Let $\tilde{\eta}^a$ be a Killing vector field on the physical space-time manifold $(\tilde{M}, \tilde{g}_{ab})$, i.e. which satisfies the equation $\mathcal{L}_{\tilde{\eta}}\tilde{g}_{ab} = 0$. From now on $\mathcal{L}$ will denote the Lie derivative of the corresponding quantity (in the previous case it was taken with respect to the vector field $\tilde{\eta}^a$). The vector field $\tilde{\eta}^a$ has a unique, smooth extension η^a (with $\tilde{\eta}^a = \eta^a|_{\tilde{M}}$) to the conformal space-time manifold (M, g_{ab}, Ω), where M, Ω and $g_{ab} = \Omega^2 \tilde{g}_{ab}$ denote the extended, unphysical space-time, the conformal factor (which vanishes on null infinity) and the unphysical metric, respectively. The vector field η^a is tangential to null infinity $\mathcal{J}$ (the symbol $\mathcal{J}$ will denote in this paper both of future and past null infinity, i.e. $\mathcal{J}^+$ and $\mathcal{J}^-$, respectively), i.e. $\eta(\Omega)|_{\mathcal{J}} = 0$ is satisfied [9]. On (M, g_{ab}, Ω) the vector field η^a is a conformal Killing vector field, i.e. equation

$$\mathcal{L}_\eta g_{ab} = \nabla_a \eta_b + \nabla_b \eta_a = \eta(\omega)\, g_{ab} \quad \text{with} \quad \omega = \ln(\Omega^2) \tag{2.1}$$

is satisfied,[1] where ∇_a denotes the Levi-Civita differential operator corresponding to the conformal metric g_{ab}. Substituting equation (2.1) into the definition of the curvature tensor $\nabla_a \nabla_b \eta_c - \nabla_b \nabla_a \eta_c = R_{abc}{}^f \eta_f$, then contracting with the unphysical metric g^{ab}, we get the equation

$$\Box \eta_a + \nabla_a \eta(\omega) + R_a{}^f \eta_f = 0, \tag{2.2}$$

where we introduced the D'Alambert operator $\Box := \nabla_f \nabla^f$. The above equation is satisfied by all conformal Killing vector fields. Moreover, equation (2.2) is a linear wave equation, so prescribing suitable initial conditions we can formulate well defined initial value problems for an arbitrary vector field η_a satisfying the above equation. We will show in the following that vector fields satisfying equation (2.2), provided additionally with appropriate initial data, are indeed satisfying equation (2.1) on the unphysical space-time, i.e. they are proper Killing vector fields in the physical space-time.

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[1]It is worth emphasizing that $\eta(\omega)$ is a regular expression even on null infinity, where Ω vanishes.

Differentiating equation (2.2) we get after some algebra the expression

$$0 = \Box \nabla_a \eta_b + (\nabla_a R_b{}^f - \nabla_b R_a{}^f + \nabla^f R_{ab})\eta_f + 2\, R_a{}^e{}_b{}^f \nabla_e \eta_f - R_a{}^f \nabla_f \eta_b + R_b{}^f \nabla_a \eta_f + \nabla_a \nabla_b \eta(\omega). \tag{2.3}$$

Introducing the tensor field

$$C_{ab} = \nabla_a \eta_b + \nabla_b \eta_a - \eta(\omega)g_{ab} \tag{2.4}$$

equation (2.3) implies the expression

$$0 = \Box C_{ab} + 2\, R_a{}^e{}_b{}^f C_{ef} - R_a{}^f C_{fb} - R_b{}^f C_{fa} + g_{ab}\Box\eta(\omega) + 2\,\mathcal{L}_\eta R_{ab} + 2\,\nabla_a\nabla_b\eta(\omega). \tag{2.5}$$

The last term of the previous equation can be rewritten as

$$2\,\nabla_a\nabla_b\eta(\omega) = (\nabla_a C_{bf} + \nabla_b C_{af} - \nabla_f C_{ab})\nabla^f\omega + \mathcal{L}_\eta(\nabla_a\omega\nabla_b\omega + 2\,\nabla_a\nabla_b\omega) - g_{ab}(\nabla^f\omega)\mathcal{L}_\eta\nabla_f\omega, \tag{2.6}$$

while for $\Box\eta(\omega)$ we can derive the equation

$$\Box\eta(\omega) = C_{ab}\nabla^a\nabla^b\omega - \frac{1}{2}C_{ab}(\nabla^a\omega)\nabla^b\omega + \frac{1}{3}\eta(\omega)(\tilde{R} - R) + \frac{1}{3}\mathcal{L}_\eta(\tilde{R} - R), \tag{2.7}$$

where we introduced $\tilde{R} = g^{ef}\tilde{R}_{ef}$, while R denotes the curvature scalar of the unphysical space-time. Deriving the above equations we used several times that η_a is a solution of the wave equation (2.2). Substituting the formulae (2.6) and (2.7) into (2.5) and utilizing that

$$2\,\mathcal{L}_\eta(R_{ab} + \nabla_a\nabla_b\omega) = \mathcal{L}_\eta\Big\{2\,\tilde{R}_{ab} - \frac{1}{3}g_{ab}(\tilde{R} - R) + \frac{1}{2}g_{ab}(\nabla_f\omega)\nabla^f\omega - (\nabla_a\omega)\nabla_b\omega\Big\}, \tag{2.8}$$

where we have used the conformal transformation formula

$$R_{ab} = \tilde{R}_{ab} + \frac{3}{\Omega^2}g_{ab}(\nabla_f\Omega)\nabla^f\Omega - \frac{1}{\Omega}\Big\{2\,\nabla_a\nabla_b\Omega + g_{ab}\nabla_f\nabla^f\Omega\Big\} \tag{2.9}$$

for the Ricci tensor, we arrive at our evolution equation

$$\begin{aligned} 0 =& \Box C_{ab} + 2\, R_a{}^e{}_b{}^f C_{ef} - R_a{}^f C_{fb} - R_b{}^f C_{fa} + \big\{\nabla_a C_{bf} + \nabla_b C_{af} - \nabla_f C_{ab}\big\}\nabla^f\omega \\ &+ \big\{\tfrac{1}{2}(\nabla_f\omega)\nabla^f\omega - \tfrac{1}{3}(\tilde{R} - R)\big\}C_{ab} - g_{ab}\big\{(\nabla^e\omega)\nabla^f\omega - \nabla^e\nabla^f\omega\big\}C_{ef} + 2\mathcal{L}_\eta\tilde{R}_{ab} \end{aligned} \tag{2.10}$$

for the tensor field C_{ab}.

3   Vacuum space-times

Equation (2.10) is a second order, linear, hyperbolic partial differential equation for the tensor field C_{ab}, in vacuum ($\tilde{R}_{ab} = 0$) it is additionally homogeneous. This means that the following assertion is just a simple consequence of the general existence and uniqueness theorems for wave equations (cf. Ref. [19]).

THEOREM 3.1 *Let (M, g_{ab}, Ω) denote some conformally compactified, asymptotically flat vacuum space-time. If the vector field η_a is a nontrivial solution of the evolution equation (2.2), furthermore the tensor field C_{ab} vanishes on the initial surfaces (hypersurface) of the considered asymptotic characteristic (hyperboloidal) initial value problem, then $\tilde{\eta}^a = \eta^a|_{\tilde{M}}$ is a Killing vector field on the considered region of the physical space-time.*

It is worth noting that the regularity of the principal part of (2.10) allows the use of the standard energy estimate methods for proving the uniqueness of the $\{C_{ab} \equiv 0\}$ (i.e. η^a is a conformal Killing vector) solution.

Now we turn to the more general case where also matter fields are present in the space-time. First we will derive some general results, then finally we discuss the cases of massless scalar and electro-magnetic field.

4 Space-times with matter fields

We start with some general assumption on the matter fields admitted in space-times which will be discussed in our following studies. We will suppose that the energy-impulse tensor has the structure

$$\tilde{T}_{ab} = \tilde{T}_{ab}(\tilde{\Phi}_A^{(i)}, \tilde{\nabla}_e \tilde{\Phi}_A^{(i)}, \tilde{g}_{ef}), \tag{4.1}$$

i.e. it depends on some matter fields $\tilde{\Phi}_A^{(i)} \equiv \tilde{\Phi}_{a_1\ldots a_n}^{(i)}$ (where capital indexes like $''A, B, \ldots''$ are multi indexes denoting a collection $''a_1 a_2 \ldots''$ of covariant indexes, while $''i''$ is just to label the several matter fields), on their first covariant derivatives and on the physical metric. The fields $\tilde{\Phi}_A^{(i)}$ are supposed to have regular limits at null infinity $\mathcal{J}$, so they can be extended smoothly to well-defined tensor fields $\Phi_A^{(i)}$ on the unphysical space-time where $\tilde{\Phi}_A^{(i)} = \Phi_A^{(i)}|_{\tilde{M}}$ is satisfied. Supposing that the Einstein equation holds, we get an expression similar to (4.1) for the physical Ricci tensor[2]

$$\tilde{R}_{ab} = \tilde{R}_{ab}(\tilde{\Phi}_A^{(i)}, \tilde{\nabla}_e \tilde{\Phi}_A^{(i)}, \tilde{g}_{ef}). \tag{4.2}$$

Equation (2.10) contains the Lie derivative of the physical Ricci tensor

$$\mathcal{L}_\eta \tilde{R}_{ab} = \sum_i \frac{\partial \tilde{R}_{ab}}{\partial \tilde{\Phi}_A^{(i)}} \mathcal{L}_\eta \tilde{\Phi}_A^{(i)} + \sum_i \frac{\partial \tilde{R}_{ab}}{\partial \tilde{\nabla}_e \tilde{\Phi}_A^{(i)}} \mathcal{L}_\eta \tilde{\nabla}_e \tilde{\Phi}_A^{(i)} + \frac{\partial \tilde{R}_{ab}}{\partial \tilde{g}_{ef}} \mathcal{L}_\eta \tilde{g}_{ef}. \tag{4.3}$$

Like the matter fields $\tilde{\Phi}_A^{(i)}$, their Lie derivatives $\mathcal{L}_\eta \tilde{\Phi}_A^{(i)}$ are also well-defined tensor fields on the whole unphysical space-time, more precisely $\mathcal{L}_\eta \tilde{\Phi}_A^{(i)} = \mathcal{L}_\eta \Phi_A^{(i)}|_{\tilde{M}}$ is satisfied. This means that we can omit the tildes also from the $\Phi_A^{(i)}$-s and the Lie derivative appearing in the second term of equation (4.3) can be rewritten as

$$\mathcal{L}_\eta(\tilde{\nabla}_e \Phi_{a_1\ldots a_n}^{(i)}) = \tilde{\nabla}_e \mathcal{L}_\eta \Phi_{a_1\ldots a_n}^{(i)} - \sum_{j=1}^n (\tilde{\nabla}\mathcal{L}_\eta \tilde{g})_{ea_j f} \tilde{g}^{fh} \Phi_{a_1\ldots h\ldots a_n}^{(i)}, \tag{4.4}$$

where we used the abbreviation

$$(\tilde{\nabla}\mathcal{L}_\eta \tilde{g})_{ea_j f} = \frac{1}{2}\left(\tilde{\nabla}_e \mathcal{L}_\eta \tilde{g}_{a_j f} + \tilde{\nabla}_{a_j} \mathcal{L}_\eta \tilde{g}_{ef} - \tilde{\nabla}_f \mathcal{L}_\eta \tilde{g}_{ea_j}\right). \tag{4.5}$$

Both of the above equations contain the Levi-Civita differential operator $\tilde{\nabla}$ induced by the physical metric $\tilde{g}_{ab}$. In order to extend these expressions into the unphysical space-time first we have to change to the operators ∇ induced by the conformal metric g_{ab}, i.e. we have to apply the conformal transformations

$$\begin{aligned}
\tilde{\nabla}_e \mathcal{L}_\eta \Phi_{a_1\ldots a_n}^{(i)} &= \nabla_e \mathcal{L}_\eta \Phi_{a_1\ldots a_n}^{(i)} + \sum_{j=1}^n \hat{\Gamma}_{ea_j}{}^f \mathcal{L}_\eta \Phi_{a_1\ldots f\ldots a_n}^{(i)}, \\
\tilde{\nabla}_e \mathcal{L}_\eta \tilde{g}_{ab} &= \tfrac{1}{\Omega^2} \nabla_e C_{ab} + \tfrac{1}{\Omega^2}\left(\hat{\Gamma}_{ea}{}^f C_{fb} + \hat{\Gamma}_{eb}{}^f C_{af} - \tfrac{2}{\Omega} C_{ab}\nabla_e\Omega\right),
\end{aligned} \tag{4.6}$$

where we used the symbols

$$\hat{\Gamma}_{ab}{}^c = \frac{2}{\Omega}\left(\delta_{(a}^c \nabla_{b)}\Omega - \frac{1}{2} g_{ab} g^{cd}\nabla_d \Omega\right). \tag{4.7}$$

We applied also the relation

$$\mathcal{L}_\eta \tilde{g}_{ab} = \frac{C_{ab}}{\Omega^2} \tag{4.8}$$

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[2]It is worth remarking, that we could have started, just like in [17], with imposing the conditions (4.2) and (4.10), the analysis itself is independent of the exact form of the Einstein equation.

which follows directly from the definition (2.4) of the tensor field C_{ab}. Substituting all the above-quoted formulae into expression (4.3) we get an equation with the structure

$$\mathcal{L}_\eta \tilde{R}_{ab} = \mathcal{A}_{ab}(\mathcal{L}_\eta \Phi_A^{(i)}) + \mathcal{B}_{ab}(\nabla_e \mathcal{L}_\eta \Phi_A^{(i)}) + \mathcal{C}_{ab}(C_{ef}) + \mathcal{D}_{ab}(\nabla_e C_{fg}), \qquad (4.9)$$

where all of $\mathcal{A}_{ab}$, $\mathcal{B}_{ab}$, $\mathcal{C}_{ab}$ and $\mathcal{D}_{ab}$ are linear, homogeneous functions in their indicated arguments. However, as we will see later on the explicit examples, some of these functions can contain also terms with negative powers of the conformal factor Ω which vanishes on null infinity. This means that also singular terms can appear on the right hand side of (4.9).

Let us suppose that the matter fields $\tilde{\Phi}_A^{(i)} = \Phi_A^{(i)}|_{\tilde{M}}$ satisfy the field equations

$$\tilde{\nabla}_a \tilde{\nabla}^a \tilde{\Phi}_A^{(i)} = F_A^{(i)}(\tilde{\Phi}_B^{(j)}, \tilde{\nabla}_e \tilde{\Phi}_B^{(j)}, \tilde{g}_{ef}), \qquad (4.10)$$

where $F_A^{(i)}$ denote some smooth functions of the indicated arguments (the matter fields are admitted to be coupled to each other). Calculating the Lie derivative of the previous equation, then performing the same transformations as above, we can derive the evolution equations

$$\Box \mathcal{L}_\eta \Phi_A^{(i)} = \mathcal{E}_A^{(i)}(\mathcal{L}_\eta \Phi_B^{(j)}) + \mathcal{F}_A^{(i)}(\nabla_e \mathcal{L}_\eta \Phi_B^{(j)}) + \mathcal{G}_A^{(i)}(C_{ef}) + \mathcal{H}_A^{(i)}(\nabla_e C_{fg}) \qquad (4.11)$$

for the Lie derivatives of the matter fields (we already performed the conformal transformations (4.6)-(4.8), as well). The functions $\mathcal{E}_A^{(i)}$, $\mathcal{F}_A^{(i)}$, $\mathcal{G}_A^{(i)}$ and $\mathcal{H}_A^{(i)}$ are linear and homogeneous in their indicated arguments. However, their regularity, like above at (4.9), depends on the concrete physical model which is investigated.

Equations (2.10) (in view of eq. (4.9)) and (4.11) compose a system of linear, homogeneous wave equations for the variables C_{ab} and $\mathcal{L}_\eta \Phi_A^{(i)}$. The right hand sides can contain terms which are singular on null infinity, however the principal parts are always regular. This admits us to apply the general theorems (see in [19]) for proving the uniqueness of the $\{C_{ab} \equiv 0, \mathcal{L}_\eta \Phi_A^{(i)} \equiv 0\}$ solution. This means that the following assertion is just a simple consequence of the general theorems.

THEOREM 4.1 *Let (M, g_{ab}, Ω) denote some conformally compactified asymptotically flat space-time containing some matter fields $\Phi_A^{(i)}$ which satisfy the evolution equations (4.10). If the vector field η_a is a nontrivial solution of the wave equation (2.2), furthermore $\mathcal{L}_\eta \Phi_A^{(i)}$ and the tensor field C_{ab} vanish on the initial surfaces (hypersurface) of the considered asymptotic characteristic (hyperboloidal) initial value problem, then $\tilde{\eta}^a = \eta^a|_{\tilde{M}}$ is a Killing vector field on the considered region of the physical space-time.*

In the appendix we will consider two examples, the massless scalar and the electro-magnetic field, in more detail. First, it is useful to write down the actual formulae where one can observe explicitly the nature of the singularities appearing on $\mathcal{J}$, caused by the $\frac{1}{\Omega}$ terms. Secondly, the field equations for the electro-magnetic field are not automatically of the form (4.10), one have to impose suitable gauge conditions in order to apply the above general results.

5 Summary

We have derived general necessary and sufficient initial conditions for Killing vector fields in the asymptotic characteristic and hyperboloidal initial value problems. These conditions are implicit in the sense that they have to be evaluated for the components of the vector field η^a. The existence of Killing vector fields depends on the existence of $\eta^a \neq 0$ solutions for the conditions derived above for the initial data. Unfortunately the evaluation of our general implicit conditions cannot be done so general as we treated the whole problem in this paper, e.g. the calculations are fundamentally different for the asymptotic characteristic and hyperboloidal initial value problems, caused by the different causal nature of the initial manifolds. However, a future work is planed for studying some applications of the introduced general assertions.

A   APPENDIX

A.1.   Massless scalar and electro-magnetic field

Let as consider a massless scalar field $\tilde{\Phi}$ on the physical space-time manifold $(\tilde{M}, \tilde{g}_{ab})$ with evolution equation

$$\tilde{\nabla}_a \tilde{\nabla}^a \tilde{\Phi} = 0, \tag{A.1}$$

and with energy-impulse tensor

$$\tilde{T}_{ab} = \tilde{\nabla}_a \tilde{\Phi} \tilde{\nabla}_b \tilde{\Phi} - \frac{1}{2} \tilde{g}_{ab} \tilde{\nabla}_f \tilde{\Phi} \tilde{\nabla}^f \tilde{\Phi}. \tag{A.2}$$

The scalar field $\tilde{\Phi}$ is supposed to have regular limit on null infinity, so it has a unique extension Φ (with $\tilde{\Phi} = \Phi|_{\tilde{M}}$) which is regular on the whole unphysical space-time (M, g_{ab}, Ω). Through the Einstein equations the Lie derivative of the physical Ricci tensor with respect to η^a is simply

$$\mathcal{L}_\eta \tilde{R}_{ab} = 8\pi \left[(\nabla_a \mathcal{L}_\eta \Phi) \nabla_b \Phi + (\nabla_a \Phi) \nabla_b \mathcal{L}_\eta \Phi \right]. \tag{A.3}$$

Here we already performed the conformal transformation, the above expression is written already in terms of the unphysical quantities. We can recognize that (A.3) is a completely regular expression on the whole conformal space-time manifold.

We can evaluate (4.11), as well. After some longer but straightforward calculations we arrive at the equation

$$0 = \Box \mathcal{L}_\eta \Phi - (\nabla^e \nabla^f \Phi) C_{ef} - g^{ef} \nabla^h \Phi \left\{ \nabla_e C_{fg} - \frac{1}{2} \nabla_g C_{ef} \right\} - 2 \frac{\nabla^e \Omega}{\Omega} \left\{ \nabla_e \mathcal{L}_\eta \Phi - (\nabla^f \Phi) C_{ef} \right\}. \tag{A.4}$$

Equations (2.10) (in view of (A.3)) and (A.4) constitute a linear, homogeneous system of wave equations for the variables C_{ab} and $\mathcal{L}_\eta \Phi$, so the following statement is just a special case of the general theorem formulated in the previous section.

PROPOSITION A.1 *Let (M, g_{ab}, Ω) denote some conformally compactified asymptotically flat space-time containing some massless scalar field in the considered region. If the vector field η_a is a nontrivial solution of the evolution equation (2.2), furthermore $\mathcal{L}_\eta \Phi$ and the tensor field C_{ab} vanish on the initial surfaces (hypersurface) of the considered asymptotic characteristic (hyperboloidal) initial value problem, then $\tilde{\eta}^a = \eta^a|_{\tilde{M}}$ is a Killing vector field on the considered region of the physical space-time.*

Now we turn to the study of the electro-magnetic field. The field equations in the physical space-time are given by

$$(^*d\tilde{F})_a = 0, \qquad \tilde{F}_{ab} = (d\tilde{A})_{ab}, \tag{A.5}$$

where d denotes the exterior differential of the corresponding quantity and the star indicates the Hodge-dual. The vector potential is considered as the restriction to $\tilde{M}$ of a $\tilde{A}_a = A_a|_{\tilde{M}}$ one-form field A_a which is regular on the whole conformally extended space-time manifold. The electro-magnetic field equations are conformally invariant, so the previous equations can be rewritten as

$$\nabla^a F_{ab} = 0, \qquad F_{ab} = \nabla_a A_b - \nabla_b A_a, \tag{A.6}$$

where we already used the conformal Levi-Civita differential operator induced by the unphysical metric g_{ab}. The energy-impulse tensor and the physical Ricci tensor, using the Einstein equations, can be written simply as

$$\tilde{T}_{ab} = F_{ae} F_{bf} \tilde{g}^{ef} - \frac{1}{4} \tilde{g}_{ab} F_{ce} F_{df} \tilde{g}^{cd} \tilde{g}^{ef}, \qquad \tilde{R}_{ab} = 8\pi \left[F_{ae} F_{bf} \tilde{g}^{ef} - \frac{1}{4} \tilde{g}_{ab} F_{ce} F_{df} \tilde{g}^{cd} \tilde{g}^{ef} \right], \tag{A.7}$$

respectively. It is easy to check that the Lie derivative (4.9) of the physical Ricci tensor now takes the form

$$\mathcal{L}_\eta \tilde{R}_{ab} = \Omega^2 \big\{ \mathcal{A}_{ab}(\mathcal{L}_\eta A_e) + \mathcal{C}_{ab}(C_{ef}) \big\}, \tag{A.8}$$

where $\mathcal{A}_{ab}$ and $\mathcal{C}_{ab}$ are some regular functions, homogeneous and isotropic in their indicated arguments.

The evaluation of (4.11) requires a bit more work. Calculating the Lie-derivative of the first equation from (A.6) after some lengthy but straightforward calculation we arrive at

$$\begin{aligned} 0 =& \Box \mathcal{L}_\eta A_a - g^{fg}\big\{ \nabla^e \nabla_f A_a - \nabla^e \nabla_a A_f \big\} C_{eg} + g^{gh} \nabla^f A_h \big\{ \nabla_g C_{af} - \nabla_f C_{ag} \big\} - R_a{}^g \mathcal{L}_\eta A_g \\ & - \nabla_a \big\{ (\mathcal{L}_\eta + \eta(\omega))(\nabla^f A_f - A(\omega)) + g^{ef} g^{gh}(\nabla_e A_g - A_e \nabla_g \omega) C_{fh} + (\nabla^f \omega) \mathcal{L}_\eta A_f \big\}. \end{aligned} \tag{A.9}$$

During the derivation of the previous equation we used only identity (4.4), equation (4.8) and the evolution equation (2.2) satisfied by the vector field η_a.

It is easy to check that in terms of the conformal quantities the Lorentz gauge condition can be rewritten as

$$\tilde{\nabla}^f \tilde{A}_f = \Omega^2 [\nabla^f A_f - A(\omega)] = 0. \tag{A.10}$$

This means that in the Lorenz gauge equation (A.9) takes a more simple form

$$\begin{aligned} 0 =& \Box \mathcal{L}_\eta A_a - g^{fg}\big\{ \nabla^e \nabla_f A_a - \nabla^e \nabla_a A_f \big\} C_{eg} - g^{gh} \nabla^f A_h \big\{ \nabla_a C_{fg} + \nabla_f C_{ag} - \nabla_g C_{af} \big\} \\ & - R_a{}^g \mathcal{L}_\eta A_g - g^{fh} g^{gk} \big\{ \nabla_a \nabla_h A_k - \nabla_a A_h \nabla_k \omega - A_h \nabla_a \nabla_k \omega \big\} C_{fg} \\ & - (\nabla_a \nabla^f \omega) \mathcal{L}_\eta A_f - (\nabla^f \omega) \nabla_a \mathcal{L}_\eta A_f. \end{aligned} \tag{A.11}$$

This expression has already the structure of (4.11). At this point we can argue like above, i.e. equations (A.11) and (2.10) constitute a linear, homogeneous system of wave equations with a unique $\mathcal{L}_\eta A_a \equiv 0$ and $C_{ab} \equiv 0$ solution. So the following statement is just a special case of the general theorem of the previous section.

PROPOSITION A.2 *Let (M, g_{ab}, Ω) denote some conformally compactified asymptotically flat space-time containing electro-magnetic field in the considered region. Let the vector potential $\tilde{A}_a$ is given in the Lorentz-gauge, i.e. $\nabla^a \tilde{A}_a = 0$ satisfied. If the vector field η_a is a nontrivial solution of the evolution equation (2.2), furthermore $\mathcal{L}_\eta A_a$ and the tensor field C_{ab} vanish on the initial surfaces (hypersurface) of the considered asymptotic characteristic (hyperboloidal) initial value problem, then $\tilde{\eta}_a = \eta_a|_{\tilde{M}}$ is a Killing vector field in the considered region of the physical space-time.*

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